Numerical study of the accuracy of the Trapezoidal time integration scheme

The following unsteady problem:

$$u_t = \triangle u + \sin t$$

with homogeneous Dirichlet boundary conditions will be solved using linear Lagrange elements and the Trapezoidal time integration method, and then the solution will be compared to the time periodic solution for large t. The time periodic solution is $\Im(Ue^{it})$, where U solves:

$$iU - \triangle U = 1$$

Two domains will be considered:

- The one-dimensional domain $[0 \quad \pi]$: In this case an exact solution exists for U, so that both the temporal and spacial discretization errors can be studied when the mesh is refined.
- An L shaped two-dimensional domain: In this case U will be found numerically, so the difference in Ue^{it} and u will only account for the temporal error. Figure 1 shows the geometry and the mesh.

Solving for u

Defining $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{F}$ as usual:

$$M_{ij} = \int \phi_i \phi_j \qquad K_{ij} = \int \nabla\phi_i \cdot \nabla\phi_j \qquad F_i = \int \phi_i$$

The following system of equations has to be solved at each time step:

$$\left[\mathbf{M} + \frac{k}{2}\mathbf{K}\right]\mathbf{U}^{n+1} = \left[\mathbf{M} - \frac{k}{2}\mathbf{K}\right]\mathbf{U}^{n} + \frac{k}{2}\mathbf{F}\left[\sin(nk) + \sin((n+1)k)\right]$$

Solving for U

Assuming $U = \sigma + i\tau$, the equation for U becomes:

$$-\triangle\sigma - \tau = 1 \tag{1}$$
$$\sigma - \triangle\tau = 0 \tag{2}$$

With $\tau = \sigma = 0$ on the boundary. Using the same FEM space for τ and σ yields the following system of equations to be solved:

$$\begin{bmatrix} \mathbf{K} & -\mathbf{M} \\ \mathbf{M} & \mathbf{K} \end{bmatrix} \begin{bmatrix} -\sigma \\ \tau \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}$$

For the one-dimensional case U can also be found analytically. As the expression is quite large, I will not include it here.

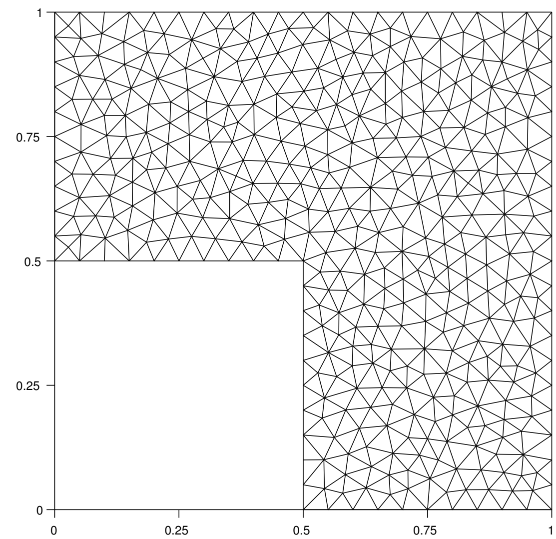

Figure 1: Mesh for the two-dimensional domain.

Results

Figure 2 shows the results for the one-dimensional domain. As expected the trapezoidal method is second order, and the reduction of the time step reduces the total error. However, after a certain point refining the time step does not reduce the error anymore, with the spacial discretiaztion errors getting dominant.

When the numerical solution is compared to the numerical time periodic solution, they both contain the spacial discretization error, which is responsible for the absence of the plateau in the trend of the computed error with respect to mesh refinement.

Figure 3 shows the numerical solution for the time steady solution $U = \sigma + i\tau$ in the L-shaped domain. At each time step the solution is found to be $\Im(Ue^{it}) = \tau \cos t + \sigma \sin t$.

Figure 4 shows the results for the two-dimensional domain. Similar to the second part of the one-dimensional case, the computed errors only show the numerical error associated with the time discretization. We clearly see that the temporal discretization method is second order.

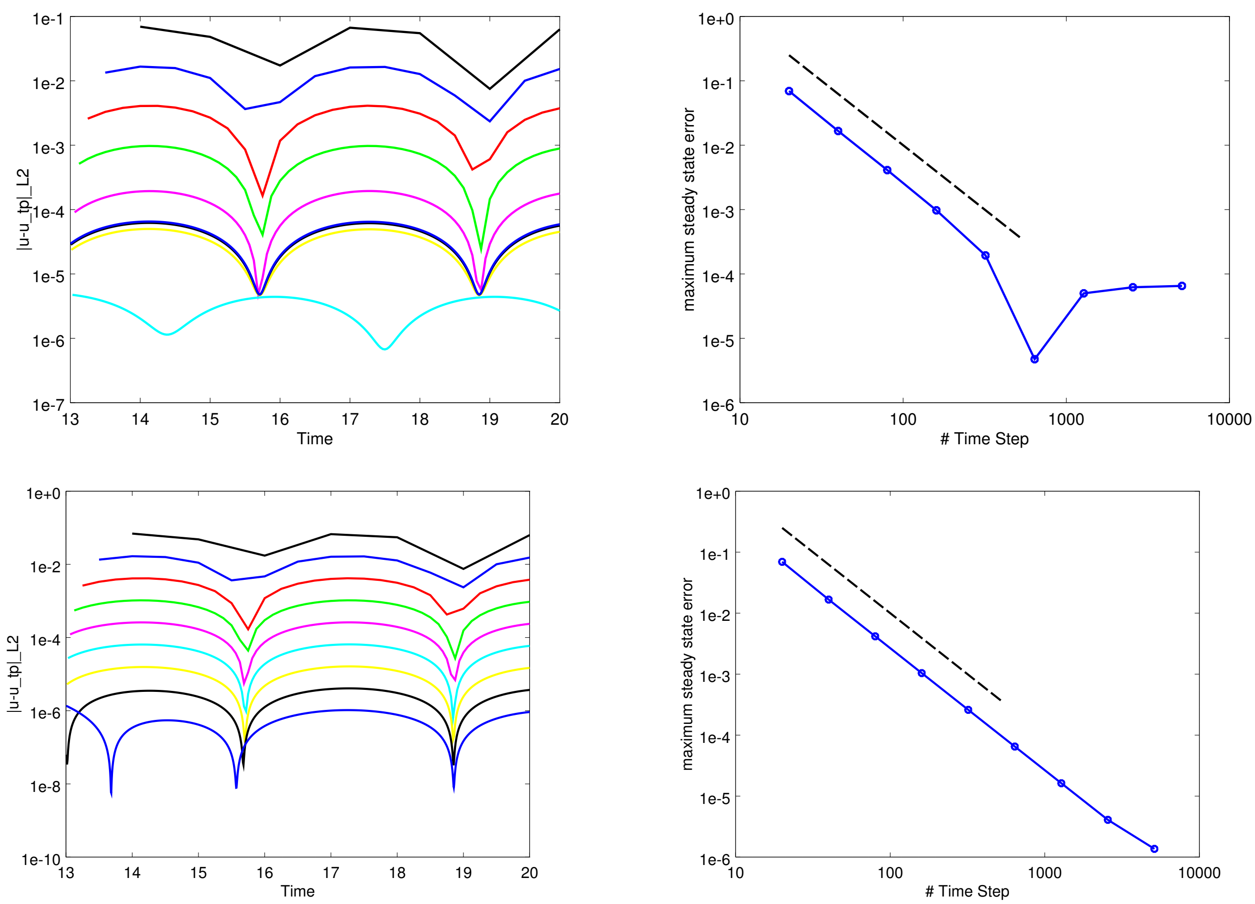

Figure 2: 1D domain: The left plots show the L_2 norm of the difference between time periodic and the numerical solution, while the right plots show the maximum of this error with respect to mesh refinement. The dashed lines show second order reduction. The exact solution for U is used to calculate the error in the upper plots. However, the numerical solution for U is used in the lower plots, i.e., they do not account for the spacial discretization error.

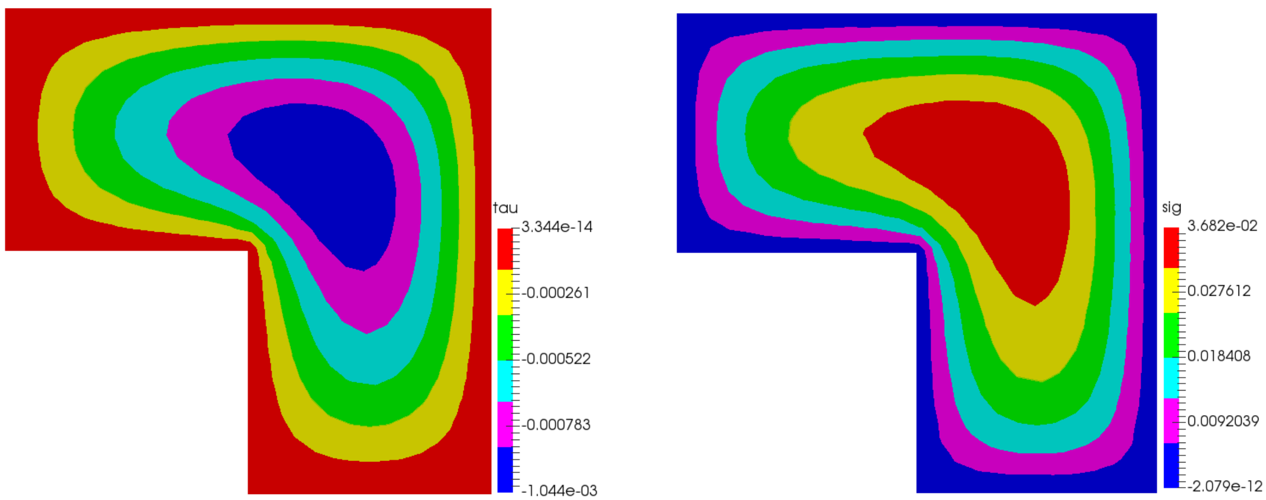

Figure 3: Numerical solution for $U = \sigma + i\tau$.

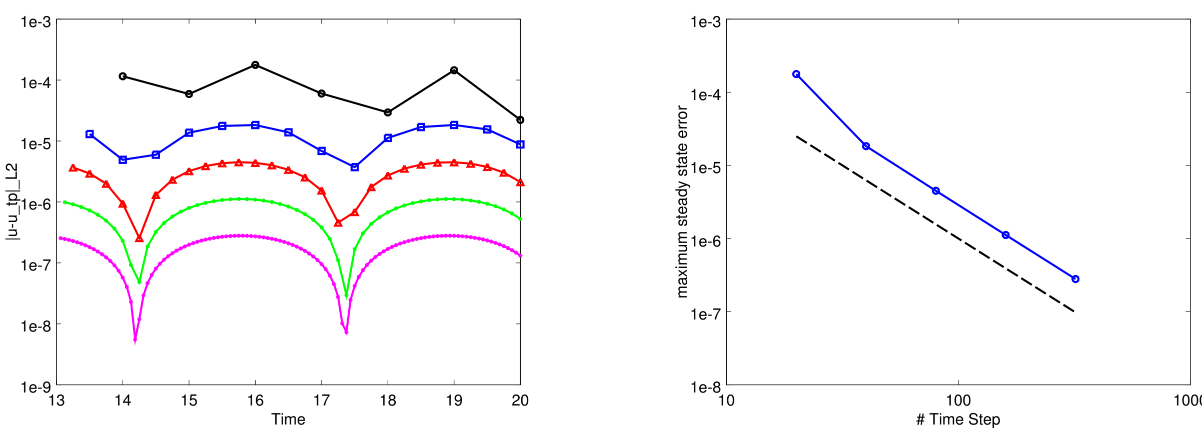

Figure 4: L shaped domain: The left plot shows the L_2 norm of the difference between the time periodic and the numerical solution, while the right plot shows the maximum of this error with respect to mesh refinement. Dashed lines represent second order reduction.