First Programming Assignment of MECH510

Shayan Hoshyari
Student #: 81382153

October 2015

Contents

1 The Problems

In this assignment the Poisson equation, Equation (1), has to be solved in the square $[0 \quad 1]\times[0 \quad 1]$, using a second order finite volume method. Both the Neumann, Equation (2), and Dirichlet, Equation (3), boundary conditions have to be considered. For convenience the boundaries $y = 0$, $x = 1$, $y = 1$, and $x = 0$ will be called bnd1, bnd2, bnd3 and bnd4 respectively.

$$T_{xx} + T_{yy} = f(x, y) \tag{1}$$

$$T_n(z) = N(z) \quad z = x, y \quad \text{on the boundary} \tag{2}$$

$$T(z) = D(z) \quad z = x, y \quad \text{on the boundary} \tag{3}$$

Two specific problems will be studied. Problem one has no source term and has an analytical solution in the form of Equation (4). The boundary conditions for this problem are $T = 0$, $T_x = 0$, $T = \cos(\pi x)$, and $T_x = 0$ for bnd1, bnd2, bnd3 and bnd4 respectively.

$$T(x, y) = \frac{\cos(\pi x)\sinh(\pi y)}{\sinh \pi} \tag{4}$$

In the second problem we use P as the unknown instead of T in accordance with the physics of the problem. This problem is more complicated and has a source term which is defined as an operator over a specified velocity field. The original definition of the source term and its simplified form are shown in Equation (5). The boundary conditions for this problem are $P_y = 0$, $P = 5 - \frac{1}{2}(1+y^2)^3$, $P = 5 - \frac{1}{2}(1+x^2)^3$ and $P_x = 0$ for bnd1, bnd2, bnd3 and bnd4 respectively.

$$
\begin{aligned}
u(x,y) &= x^3 - 3xy^2 \\
v(x,y) &= -3x^2y + y^3 \\
f(x,y) &= -(u_x^2 + 2v_xu_y + v_y^2) = -18(x^2+y^2)^2
\end{aligned}
\tag{5}
$$

2 Numerical Method

In this section the general solution procedure and the key formulas will be presented. The general numerical algorithm is given in Algorithm 1.

Algorithm 1 Numerical Solution of the Poisson equation on a unit square

Require: NI, NJ, ω, ϵ
Ensure: $\bar{T}_{i,j}$, $\|E\|_1$, $\|E\|_2$, $\|E\|_\infty$

```
Δx = 1/NI, Δy = 1/NJ
Find the coordinates of the center of control volumes: x_ij, y_ij
Find the value of source term at the center of CV's: f_ij
c = 0
while n_it < 100000 do
    Find the value of all ghost cells according to Equation (7)
    Find the new value of all cells according to Equation (6) and d_max = max(T_new − T_old)
    Print n_it and d_max.
    n_it = n_it + 1;
    if d_max < ε then
        c = 1
        Break.
    end if
end while
if c = 1 then
    State success.
    Calculate the exact mean value of solution at each CV using Equations (8) and (9).
    Calculate error at each CV.
    Calculate the Error L norms using the errors evaluated at each CV.
    Print the number of iterations and L norms.
    Output the numerical value of solution and error at each CV in vtk format.
else
    State failure.
end if
```

In this algorithm we have used the SOR method to solve our system of linear equations, which itself is derived using a finite volume method. Equation (6) shows the derived SOR iteration formula. In this equation D is a dimensionless parameter defined as $(\Delta y/\Delta x)^2$.

$$
\begin{aligned}
T_{ij}^* &= \frac{1}{2(D+1)}\left[(T_{i+1,j} + T_{i-1,j})D + T_{i,j+1} + T_{i-1,j-1} - f_{ij}\Delta y^2\right] \\
\delta_{ij} &= T_{ij}^* - T_{ij} \\
T_{ij} &:= T_{ij} + \omega\delta_{ij}
\end{aligned}
\tag{6}
$$

Table 1: The error L norms and number of iterations for problem one on a 10×10 mesh

ω	L_1	L_2	L_∞	number of iterations
1	1.2499×10^{-3}	2.4570×10^{-3}	9.0954×10^{-3}	209
1.5	1.2497×10^{-3}	2.4570×10^{-3}	9.0956×10^{-3}	92

In addition, ghost cells are used to impose the boundary conditions. The solution value at a ghost cell is evaluated using Equation (7).

$$\begin{aligned} T_{gD} &= 2D_s - T_s \\ T_{gN} &= T_s + \alpha N_s \Delta z \end{aligned} \tag{7}$$

D_s Value of T that should be enforced at the ghost cell's adjacent wall.
N_s Value of $\frac{\partial T}{\partial n}$ that should be enforced at the ghost cell's adjacent wall.
T_{gD} Value of a ghost cell at a Dirichlet boundary.
T_{gN} Value of a ghost cell at a Neumann boundary.
Δz Means Δx for bnd2 and bnd4, and means Δy for bnd1 and bnd3.
α Is equal to $+1$ for bnd2 and bnd3 and is equal to -1 for bnd4 and bnd1.

To find the error L norms of the solution, the average value of the analytical solution at each control volume is used. While using the analytical solution value at the midpoint of the CV would have been sufficient, we have used the 9 point Gauss Quadrature scheme to evaluate the analytic average values. This method is presented in Equations (8) and (9) for a function called $g(x, y)$ in a rectangle in the form $[a_1 \quad a_2] \times [b_1 \quad b_2]$.

$$\begin{aligned} x_1 &= \tfrac{1+\sqrt{3/5}}{2} a_1 + \tfrac{1-\sqrt{3/5}}{2} a_2 & y_1 &= \tfrac{1+\sqrt{3/5}}{2} b_1 + \tfrac{1-\sqrt{3/5}}{2} b_2 \\ x_2 &= \tfrac{a_1+a_2}{2} & y_2 &= \tfrac{b_1+b_2}{2} \\ x_3 &= \tfrac{1-\sqrt{3/5}}{2} a_1 + \tfrac{1+\sqrt{3/5}}{2} a_2 & y_3 &= \tfrac{1-\sqrt{3/5}}{2} b_1 + \tfrac{1+\sqrt{3/5}}{2} b_2 \end{aligned} \tag{8}$$

$$\begin{aligned} \bar{g} = \frac{1}{4} \Bigg\{ & \frac{8}{9} \cdot \frac{8}{9} g(x_2, y_2) + \\ & \frac{8}{9} \cdot \frac{1}{9} \left[g(x_2, y_1) + g(x_2, y_3) + g(x_1, y_2) + g(x_3, y_2) \right] + \\ & \frac{1}{9} \cdot \frac{1}{9} \left[g(x_1, y_1) + g(x_3, y_3) + g(x_1, y_3) + g(x_3, y_1) \right] \Bigg\} \end{aligned} \tag{9}$$

3 Results

In this section the solutions to each parts of the assignment are presented.

3.1 Point Gauss-Sidel and Point SOR

Problem one is solved on a 10×10 structured mesh, with $\omega = 1, 1.5$ and $\epsilon = 10^{-7}$. The resulting values of errors at each control volume is almost identical in both cases and is presented in Figure 1. Table 1 shows the error norms and the number of iterations for each case. As we expected, using the value of 1.5 for ω reduces the number of iterations drastically.

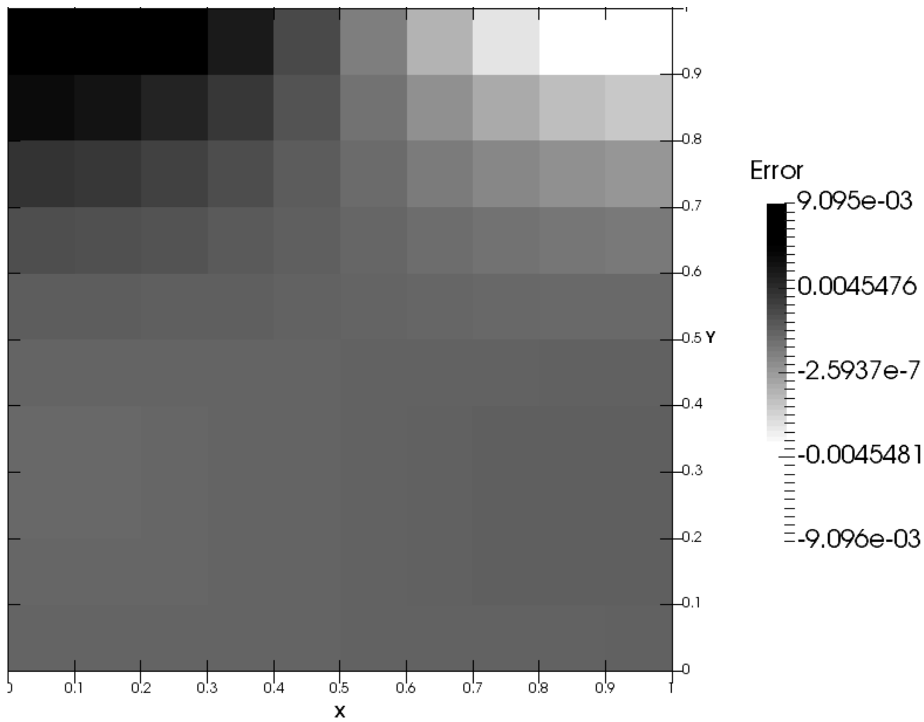

Figure 1: The value of error at each control volum in problem one, using a 10×10 mesh, $\omega = 1$ and $\epsilon = 10^{-7}$.

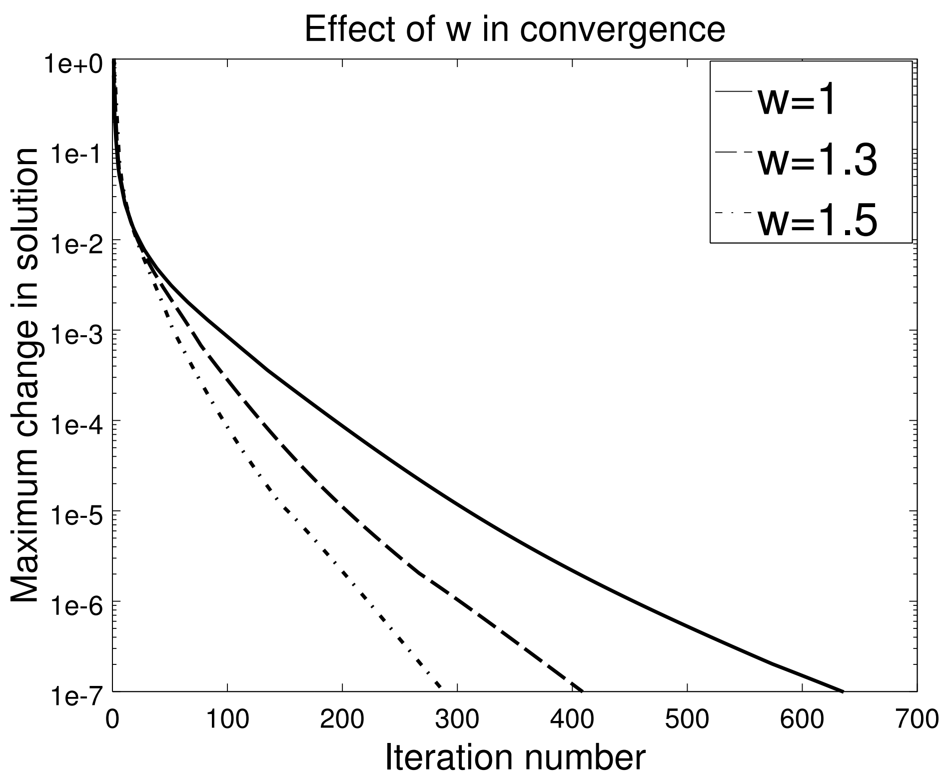

Figure 2: Maximum change in the solution per timestep for three different values of ω on a 20×20 mesh.

Table 2: Slope of $\log \|E\| - N$ diagrams

L_1	L_2	L_∞
1.9652	1.9812	1.9013

3.2 Convergence Behaviour

We will solve problem one on a 20×20 mesh for three different values of ω, i.e., 1, 1.3 and 1.5, while $\epsilon = 10^{-7}$. Then we will plot the maximum change in the solution per iteration in each case, in an effort to show how ω affects the solution convergence of our linear system of equations. This plot is shown in Figure 2 and clearly shows that, as the value of ω increases from 1 to 1.5 the maximum change in solution drops under ϵ faster, verifying the positive effect of Successive Over Relaxation.

3.3 Accuracy

The accuracy of the numerical method will be analyzed, by studying the behaviour of error L norms as the mesh gets smaller. Note that for a p-th order method we expect that:

$$\|E\| = C\,(\Delta x)^p \tag{10}$$

Taking the logarithm of both sides of Equation (10) and using the fact that $\Delta x = \frac{1}{N}$ (N is the number of control volumes in each direction), we will have:

$$log\|E\| = -p \log N - \log C \tag{11}$$

Equation (11) means that p will be the slope of the $\log \|E\| - logN$ diagram.

To study the order of accuracy our method problem one is solved on meshes with dimensions: 10×10, 20×20, 40×40, 80×80, and 160×160 and the resulting norm vs. mesh size diagrams are shown in Figure 3. The linearity of these diagrams are in agreement with our expectation of the behaviour of structured meshes. The slope of the diagrams are found using linear least square curve fitting and is presented in Table 2. Being close to 2, these slopes verify that our method is truly second order for problem one.

3.4 Pressure Problem

Problem two is solved by adding a source term to our laplace solver, as shown in Equation (6). In Figure 4 the average values of control volumes are shown on a 320×320 mesh. All the results obtained in this section are obtained by setting $\epsilon = 10^{-10}$ and $\omega = 1.5$.

To evaluate the pressure at the point $e = (^1/_2, ^1/_2)$ we interpolate the average values of the surrounding control volumes to get a second order approximation. First we label the surrounding control volumes according to Figure 5. Then using a taylor series expansion around e, we can show

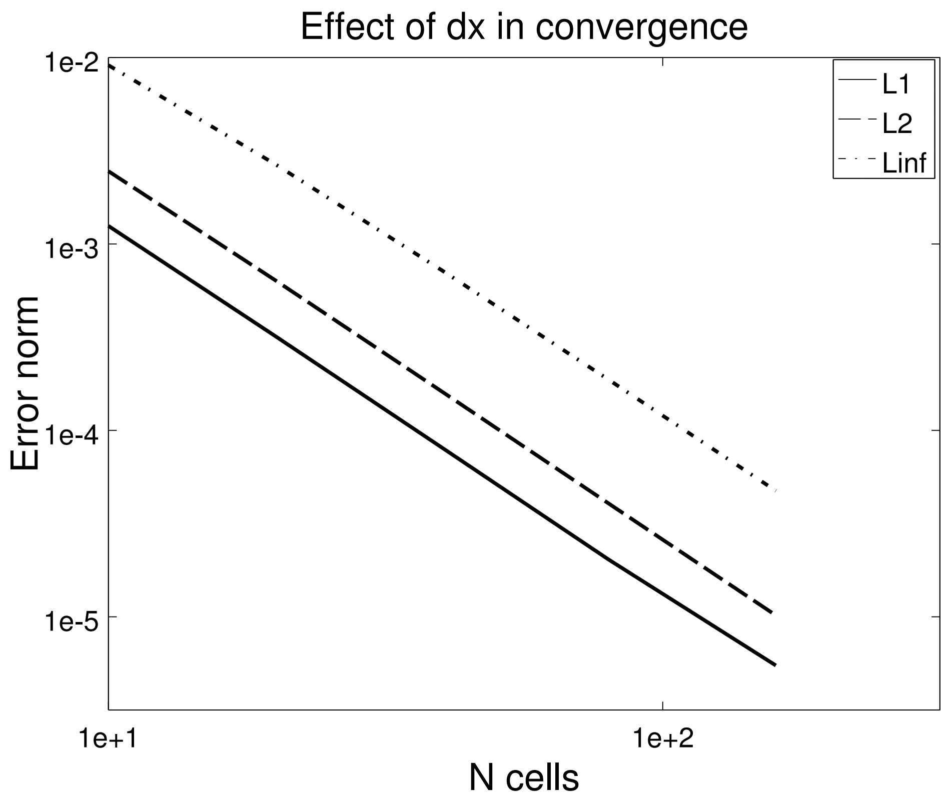

Figure 3: The norm of error vs. number of cells in each direction.

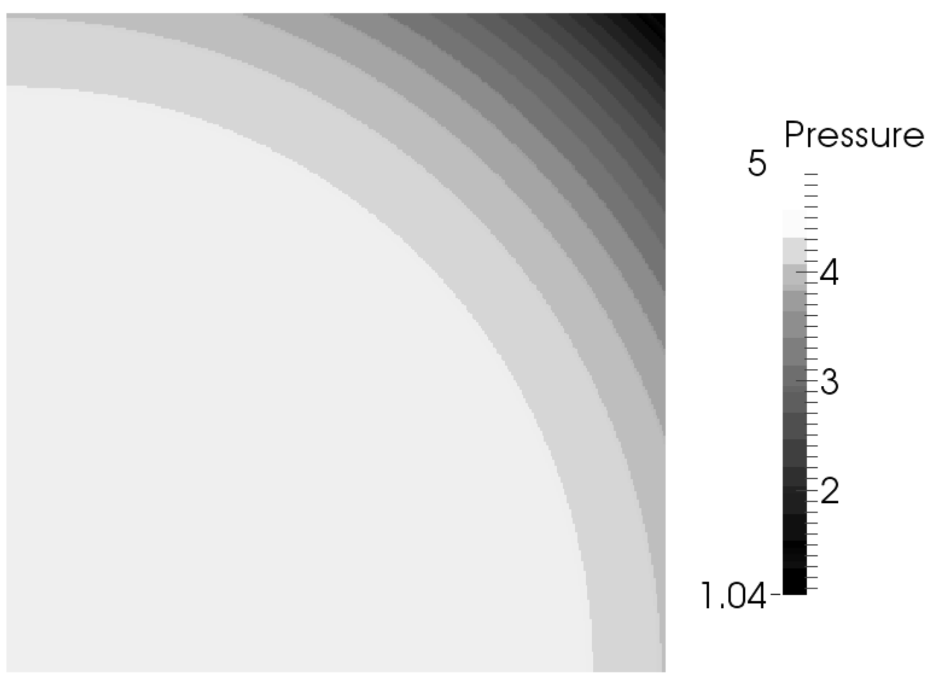

Figure 4: Solution for problem two on a 320×320 mesh

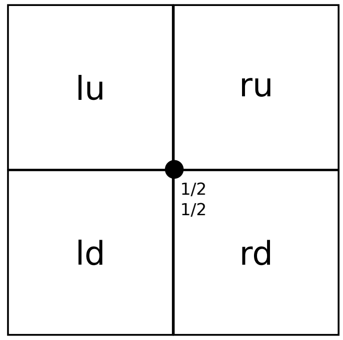

Figure 5: Labeling control volumes containing the point $^1/_2, ^1/_2$

that (all the derivatives are evaluated at e):

$$\bar{P}_{ru} = P_e + P_x\frac{\Delta x}{2} + P_y\frac{\Delta y}{2} + P_{xx}\frac{\Delta x^2}{6} + P_{xy}\frac{\Delta x\Delta y}{4} + P_{yy}\frac{\Delta y^2}{6} + \cdots \tag{12}$$

$$\bar{P}_{rd} = P_e + P_x\frac{\Delta x}{2} - P_y\frac{\Delta y}{2} + P_{xx}\frac{\Delta x^2}{6} - P_{xy}\frac{\Delta x\Delta y}{4} + P_{yy}\frac{\Delta y^2}{6} + \cdots \tag{13}$$

$$\bar{P}_{lu} = P_e - P_x\frac{\Delta x}{2} + P_y\frac{\Delta y}{2} + P_{xx}\frac{\Delta x^2}{6} + P_{xy}\frac{\Delta x\Delta y}{4} + P_{yy}\frac{\Delta y^2}{6} + \cdots \tag{14}$$

$$\bar{P}_{ld} = P_e - P_x\frac{\Delta x}{2} - P_y\frac{\Delta y}{2} + P_{xx}\frac{\Delta x^2}{6} - P_{xy}\frac{\Delta x\Delta y}{4} + P_{yy}\frac{\Delta y^2}{6} + \cdots \tag{15}$$

Now using Equations (12)-(15) we can easily show that the average of average pressure values(!) at the surrounding control volumes is a second order approximation for P_e, i.e.:

$$P_e = \frac{1}{4}\left(\bar{P}_{ru} + \bar{P}_{rd} + \bar{P}_{lu} + \bar{P}_{ld}\right) + O(\Delta x^2, \Delta y^2) \tag{16}$$

For a sequence of 40×40, 80×80, and 160×160 meshes the values of surrounding control volumes and the interpolated value for P_e are shown in Table 3. Now using Richardson Extrapolation we will find approximations to the order of accuracy and error bound of our solution. For the order of accuracy we can write:

$$p = \log_2 \frac{|P|_{M1} - P|_{M2}|}{|P|_{M2} - P|_{M3}|} = 1.995 \tag{17}$$

Equation (17) approximates the solution's order of accuracy to be 1.995. Now based on the ASME solution accuracy handout we extrapolate a more accurate value for P_e.

$$P_{\text{ext}} = \frac{2^p \times P|_{M3} - P|_{M2}}{2^p - 1} = 4.937499725036921 \tag{18}$$

Now we can calculate the error bound for our solution by using either the extrapolated pressure value or the fine mesh pressure value:

$$e_a^{M3M2} = \frac{P|_{M3} - P|_{M2}}{P_{M3}} \times 100 =: 0.00094\% \tag{19}$$

$$e_{\text{ext}}^{M3M2} = \frac{P_{\text{ext}} - P|_{M3}}{P_{\text{ext}}} \times 100 = 0.00031\% \tag{20}$$

Hopefully, we see that the final error in the pressure at the midpoint is only 0.00031%.

Table 3: Values of P_e on a series of refined meshes

Size	$\bar{P}_{lu}$	$\bar{P}_{ru}$	$\bar{P}_{ld}$	$\bar{P}_{rd}$	P_e
40×40	4.9379814815	4.9281345750	4.9468909663	4.9379814815	4.9377471261
80×80	4.9376203616	4.9328152779	4.9421910752	4.9376203616	4.9375617691
160×160	4.9375299330	4.9351568380	4.9398444345	4.9375299330	4.9375152846

4 Conclusion

In this assignment the Poisson equation was solved in a unit square with Dirichlet and Neumann boundary conditions. A cell centered finite volume method was implemented on different structured grids to discretize the PDE and SOR and Gauss-Sidel methods were used to solve the resulting system of algebraic equations.

In addition the order of accuaracy of the numerical method was studied using both the method of manufactured solutions and the Richardson Extrapolation Method. It was shown that the implemented method is second order accurate.