CPSC 530P Sensorimotor Computation
Lecture: February 14th, 2018
Prof. Dinesh Pai, Scribe: Shayan Hoshyari

Goals

Previously, we saw that a rigid motion, $x = \phi(X)$, can be expressed as:

$$x = R(q)X + P(q), \tag{1}$$

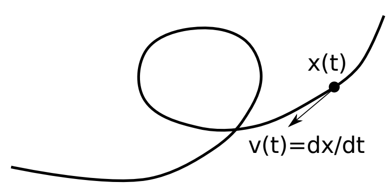
where R is a rotation matrix, P is a translation vector, and q is the vector of configurations (degrees of freedom).

In this lecture, the goal is to answer the following questions.

- How can the velocity of a rigid motion be described?
- What does the parameterization of R and P look like with respect to q?

Velocity for Rigid Motion

For a particle moving in the path $x = x(t)$, the velocity is found as $v(t) = \dot{x} = \frac{d}{dt}x(t)$. A schematic of this concept is shown in below.



For a continuous medium, the velocity has the more general form of

$$v(X,t) = \dot{x} = \frac{D}{Dt}x(X,t),$$

where D represents the material derivative, i.e. it is evaluated when the Lagrangian coordinate X is constant rather than the Eulerian coordinate x.

In order to evaluate the velocity field for the special case of the rigid motion, we recall that this type of motion has the form of Eq. (1), where the rotation matrix R must satisfy the following constraints.

$$R^T R = I, \qquad \det(R) = +1. \tag{2}$$

Although, Eq. (2) represents a series of non-linear dependent constraints, we will show that they can be greatly simplified when one looks at the velocity rather than the location. For simplicity, let us assume that there is no translation, $P = 0$. Taking the material derivative of x we would get:

$$\begin{cases} x = RX \\ R^T R = I \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{R}X \\ X = R^T x \end{cases} \rightarrow \dot{x} = \dot{R}R^T x, \tag{3}$$

where $[\omega] = \dot{R}R^T$ is known as the angular velocity matrix.

Lets us know transfer the constraints on R to $[\omega]$:

$$RR^T = I \Rightarrow \dot{R}R^T + R\dot{R}^T = 0 \Rightarrow [\omega] = -[\omega]^T.$$

This means that any skew-symmetric matrix can compactly describe the velocity of a rotational motion. Also, we can see that in n dimensions $\frac{n(n-1)}{2}$ degrees of freedom are required to describe an angular velocity matrix (1, 3, and 6 values for two, three, and four dimensions, respectively). As a special case $[\omega]$ is represented as following in two and three dimensions:

$$[\omega]_{2D} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}, \qquad [\omega]_{3D} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_z & 0 \end{bmatrix}$$

In 3-D, the angular velocity vector ω is defined as $(\omega_x, \omega_y, \omega_z)$. We can observe that $[\omega]x = \omega \times x$, where $\times$ represents the vector cross product. Therefore, the "[]" operator can be thought of as change of format from $\mathcal{R}^3$ to $\mathcal{R}^9$, transforming the angular velocity vector to the angular velocity matrix. This simplification, however, does not generalize to higher dimensions.

Having defined $[\omega]$, we now have the machinery to parameterize rotation.

Parameterizing Rotation

To parameterize rotation, we start by recalling Eq. (3). This equation represents a differential equation for position x. For a fixed $[\omega]$, this equation can be solved using our knowledge of matrix exponential:

$$x = \exp([\omega]t)x(t = 0) = \exp([\omega]t)X. \tag{4}$$

We can assume that any rotation R has happened during some imaginary time $t \in [0, 1]$ in the form of $R(t) = Rt$, which gives $[\omega] = RR^T$. Then, plugging $t = 1$ in Eq. (4) this rotation can be written as $R = \exp([\omega])$. Thus, we have worked out the inverse of Eq. (3) and expressed the rotation in terms of the angular velocity. Schematically:

$$\text{Finite rotation} \underset{\text{Log (equivalently } RR^T)}{\overset{\text{Exp}}{\leftrightarrows}} \text{Skew-symmetric matrix}$$

This gives us a nice way to parameterize finite rotations by only using a skew-symmetric matrix. This parameterization simplifies to a vector in three dimensions with the direction of the vector representing the axis of rotations and its magnitude the angle of rotation.

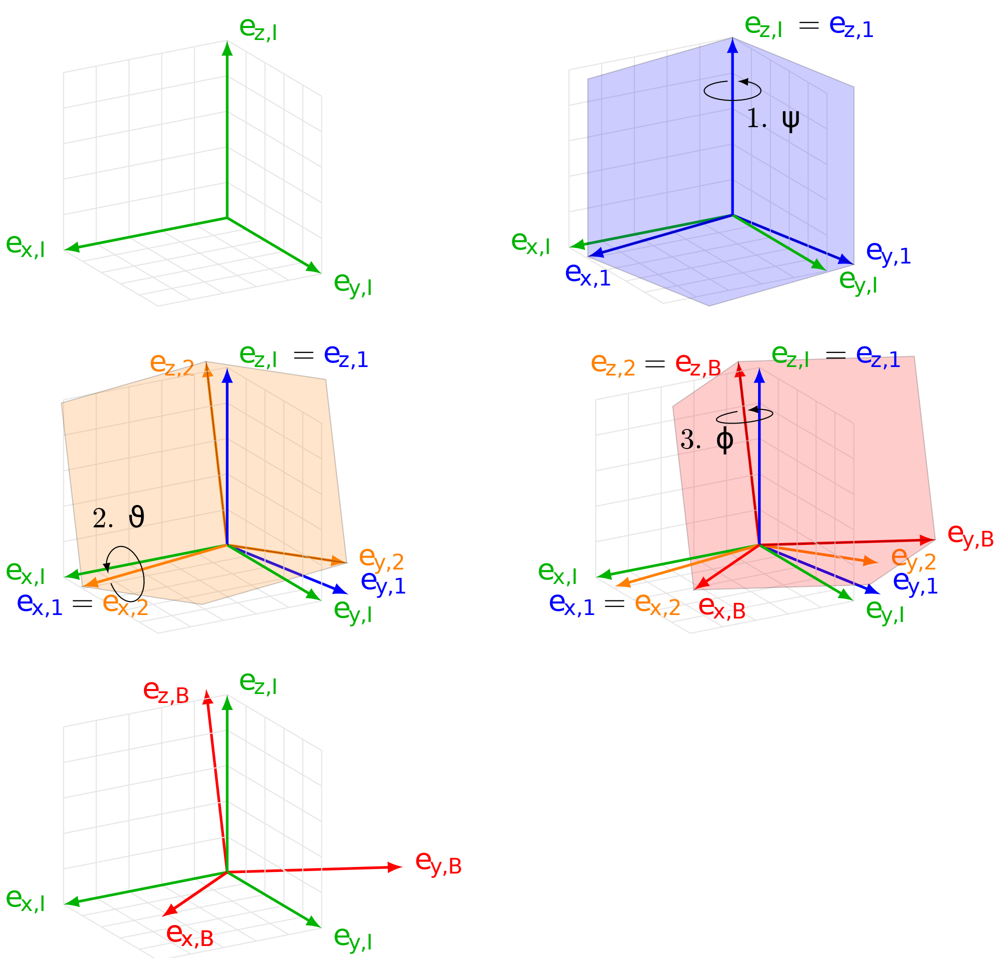
With the presented machinery, parameterizing a rigid motion in three dimension boils down to selecting three degrees of freedom $\theta = (q_1, q_2, q_3)$ that can represent the rotation part of the motion as $\exp([\theta])$, and another three degrees of freedom (q_4, q_5, q_6) that can be the coordinates of the translation vector p.

Other ways to parameterize rotation

Complex numbers In 2-D, a rotation can be represented using complex numbers. A vector $x = (x_1, x_2)$ can be represented by the complex number $x_1 + ix_2$, and a rotation equal to θ can be presented as $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$. This is equivalent to associating the number 1 to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the number i to the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Unit quaternions In 3-D a rotation can be represented using unit quaternions (four scalars plus a constraint). This is not quite surprising since quaternions are somewhat a generalization of complex numbers (tuples of two real numbers) to tuples of four real numbers.

Euler angles A 3-D rotation can be expressed in terms of Euler angles. First, a rotation of angle ψ (or α) around the z axis, then a rotation of angle θ (or β) around the new x axis, and finally a rotation of angle ϕ (or γ) around the new z axis. A schematic is shown below.



Other systems of angles Other systems include roll-yaw-pitch and Fick angles.