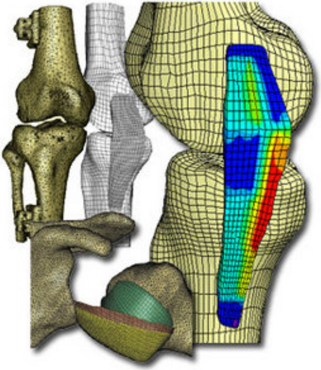
Eulerian Simulation of Large Deformations

Shayan Hoshyari

April, 2018

Some Applications

1. Biomechanical Engineering



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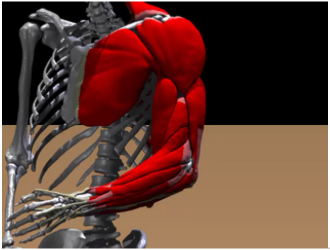
2. Muscle Animation



Some Applications

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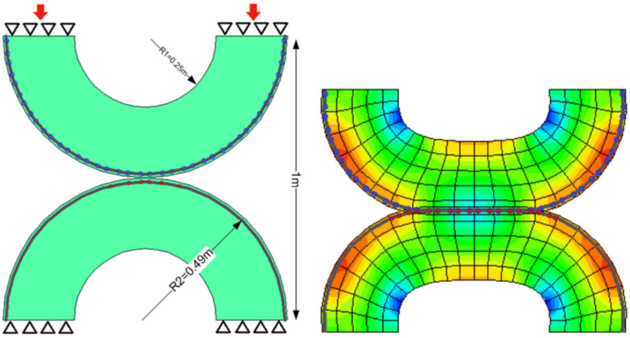
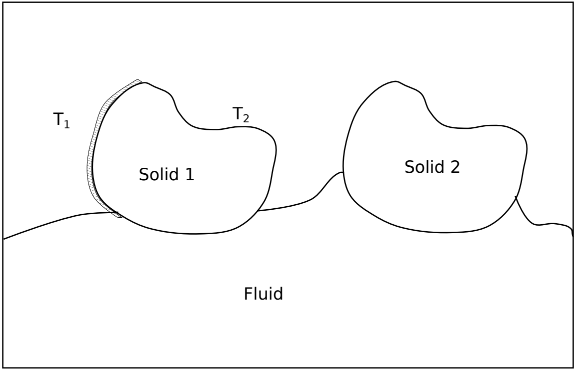
2. Muscle Animation
3. Structural Engineering



A very general problem



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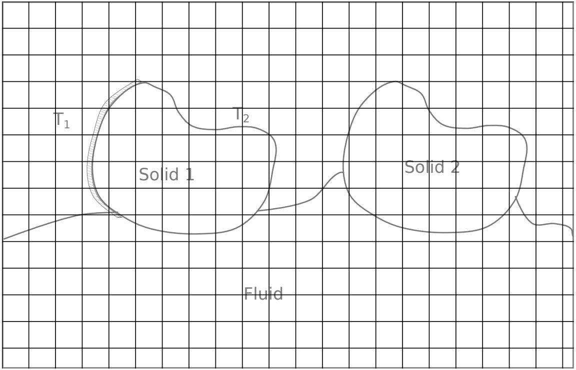

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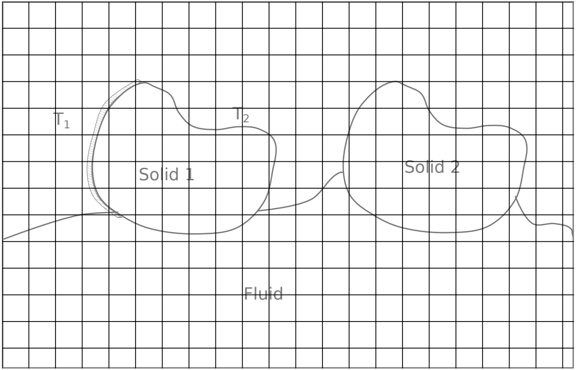

1. Fluid variables: level set ϕ, pressure p, velocity v, (density ρ)

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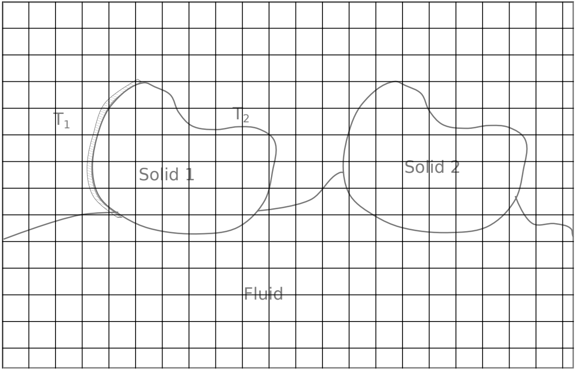

1. Fluid variables: level set ϕ, pressure p, velocity v, (density ρ)
2. Solid variables: velocity v, reference map ξ

A more specific problem

Eulerian Solid Simulation with Contact

D. I. Levin , J. Litven, G. L. Jones, S. Sueda, D. K. Pai, Siggraph 2011

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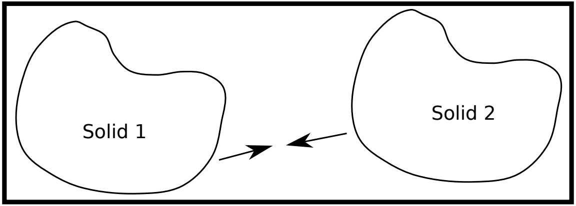

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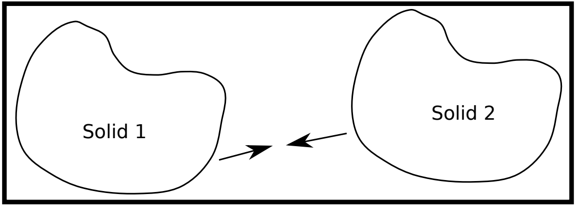

- No clamped boundaries.

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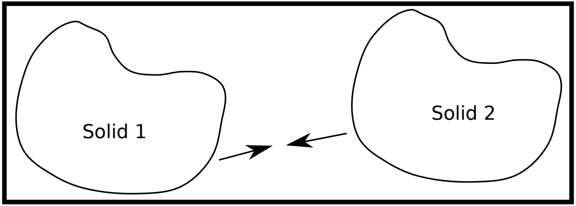

- No clamped boundaries.
- Frictionless contact.

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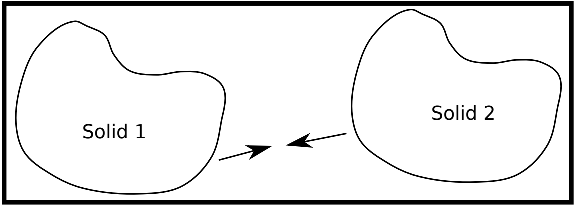

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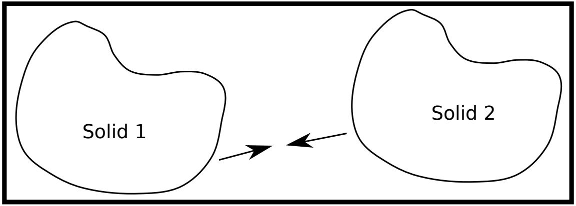

- No clamped boundaries.
- Frictionless contact.
- Large deformations.
- This presentation: formulation in 1-D.

Equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho v\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho v^T\\ \rho v v^T-\sigma\end{bmatrix}=\begin{bmatrix}0\\ \rho g\end{bmatrix}$$

- Conservation of mass and momentum.

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Equations

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Equations

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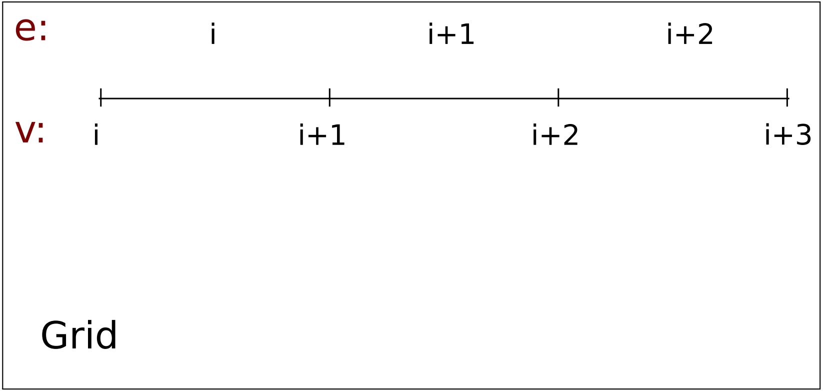
- In 1-D: $\sigma = \kappa(\frac{1}{\xi_x} - 1)$

Discretization



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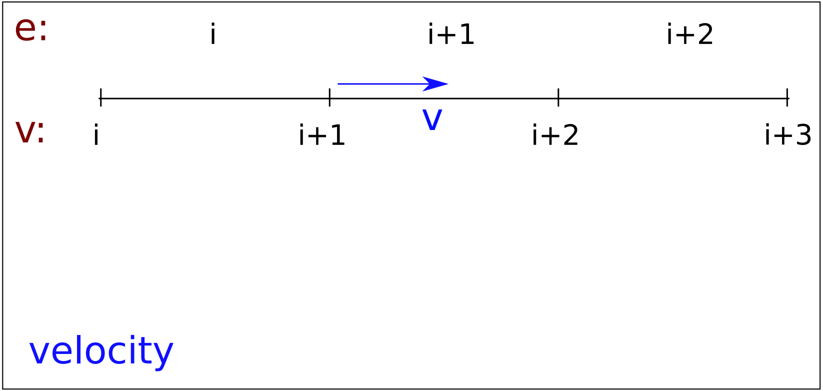

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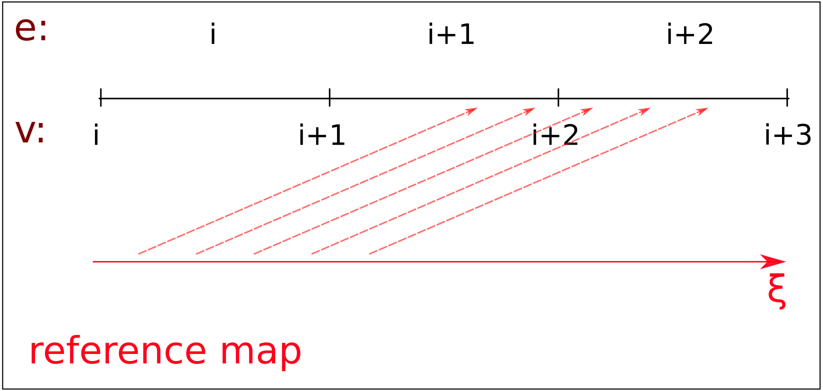

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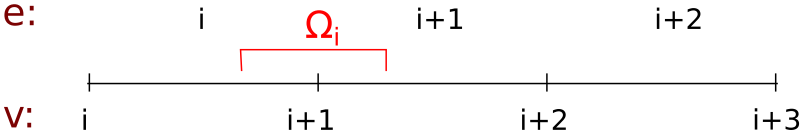
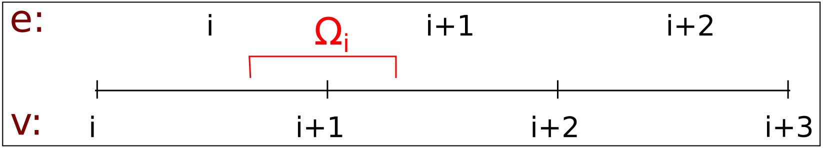


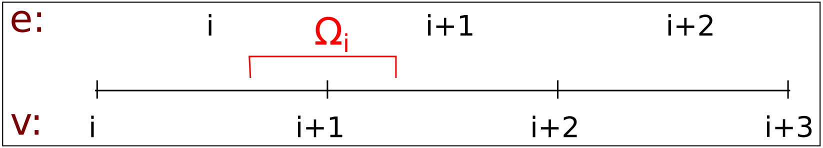
control volume

Discretization



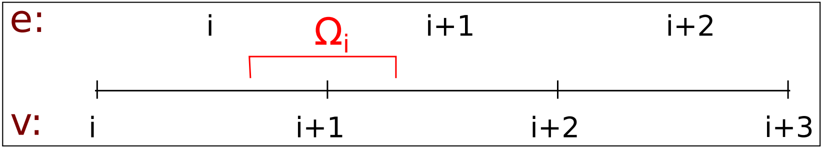
$$
\begin{cases}
\rho v_t = -\rho v v_x + \sigma_x + \rho g \\
\xi_t = -v \xi_x
\end{cases}
$$

Discretization



$$\begin{cases} \int_{\Omega_i} \left(\rho v_t = -\rho v v_x + \sigma_x + \rho g\right) \\ \xi_t = -v\xi_x \end{cases}$$

Discretization



$$\begin{cases} m_i \frac{dv_i}{dt} = -m_i (v v_x)_i + (\sigma_{i+1/2} - \sigma_{i-1/2}) + m_i g + O(\Delta x) \\ \frac{d\xi_i}{dt} = -(v \xi_x)_i \end{cases}$$

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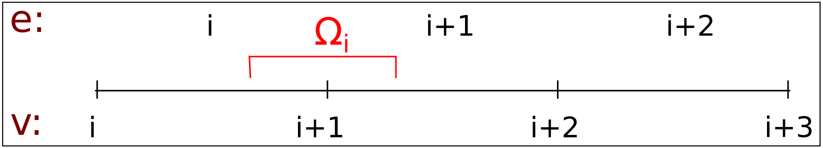

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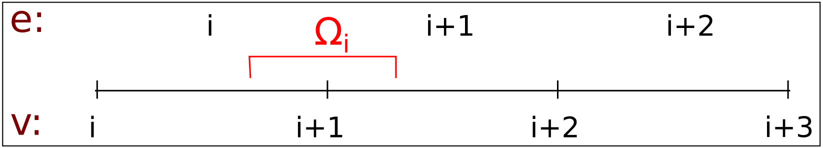

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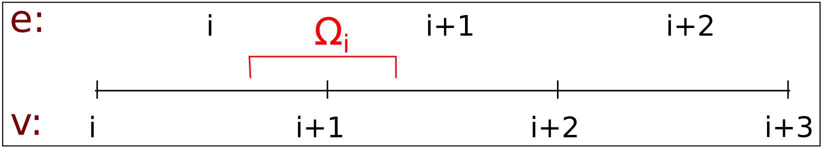
- $(vv_x)_i = \max(v_i, 0)\frac{v_i - v_{i-1}}{\Delta x} + \min(v_i, 0)\frac{v_{i+1} - v_i}{\Delta x}$

Discretization



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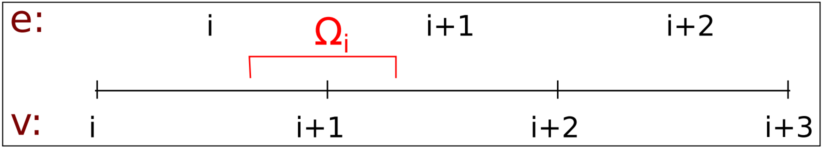

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- $\sigma_{i+1/2} = \begin{cases} \sigma((\xi_x)_{i+1/2}) & x_{i+1/2} \text{ inside solid} \\ 0 & \text{otherwise} \end{cases}$

Discretization—Continued

System of ODEs

$$\boldsymbol{M}\mathbf{v}_t = \mathbf{R}_v(\mathbf{v}, \boldsymbol{\xi})$$
$$\boldsymbol{\xi}_t = \mathbf{R}_\xi(\mathbf{v}, \boldsymbol{\xi})$$

Discretization—Continued

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Explicit Euler

- Find $\boldsymbol{M}$

Discretization—Continued

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- $\boldsymbol{M}\mathbf{v}^{(t+\Delta t)} = \boldsymbol{M}\mathbf{v}^{(t)} + \Delta t \mathbf{R}_v(\mathbf{v}^{(t)}, \boldsymbol{\xi}^{(t)}) = \mathbf{R}_{v*}(\mathbf{v}^{(t)}, \boldsymbol{\xi}^{(t)})$

Discretization—Continued

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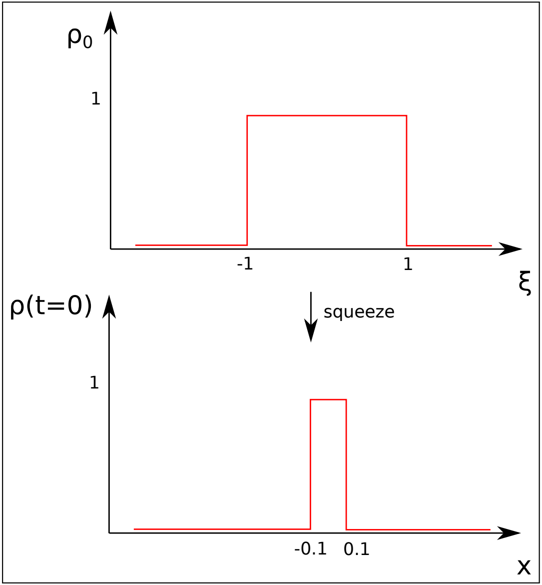
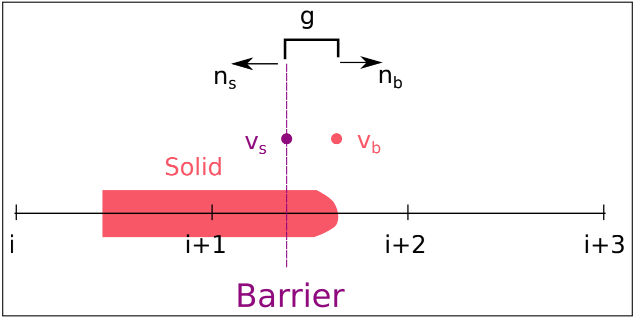
Squeezing an Object



$$v(x, t = 0) = 0$$
$$\xi(x, t = 0) = 10x$$

Solution: Link

Collision



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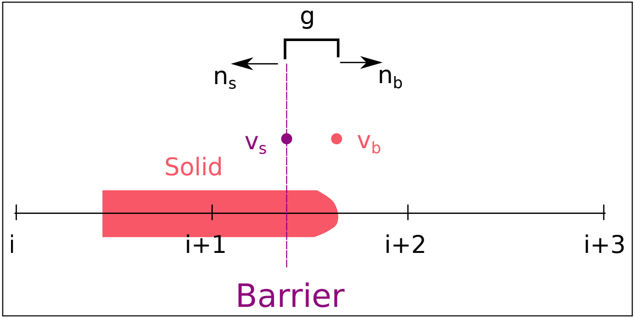
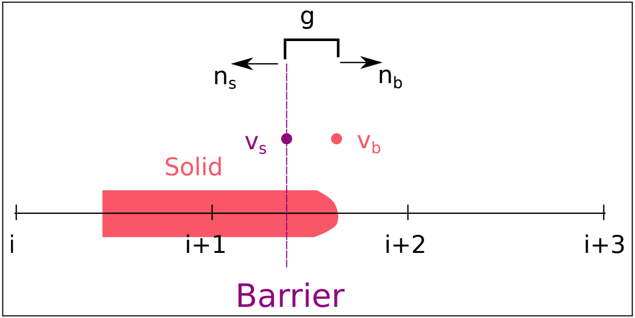


- Constraint: $g_t = v_b \cdot n_b + v_s \cdot n_s \leq 0$

Collision



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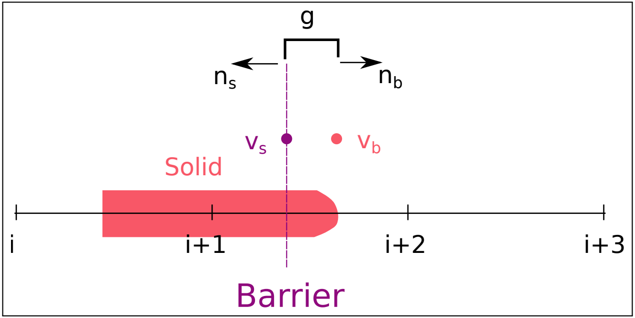

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- Old time advance: $\boldsymbol{M}\mathbf{v}^{(t+\Delta t)} = \mathbf{R}_{v*}(\mathbf{v}^{(t)}, \xi^{(t)})$

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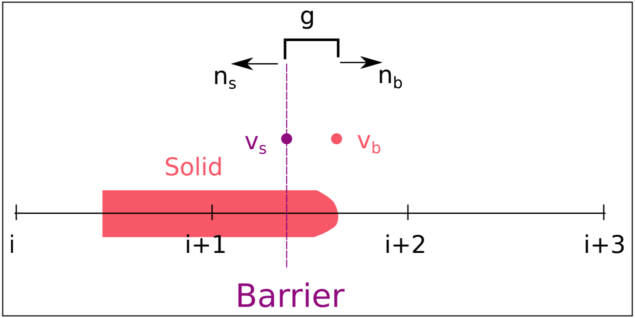

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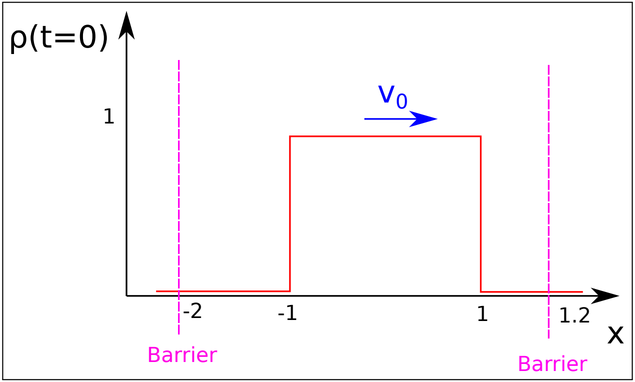
- Matrix form: $\boldsymbol{J}\mathbf{v} \leq \mathbf{b}$
- Old time advance: $\boldsymbol{M}\mathbf{v}^{(t+\Delta t)} = \mathbf{R}_{v*}(\mathbf{v}^{(t)}, \xi^{(t)})$
- New time advance:
  $\mathbf{v}^{(t+\Delta t)} = \arg\min_{\mathbf{v}}(\frac{1}{2}\mathbf{v}^T\boldsymbol{M}\mathbf{v} + \mathbf{R}_{v*}^T\mathbf{v})$, subject to $\boldsymbol{J}\mathbf{v} \leq \mathbf{b}$

Collision—Example



$v(x, t = 0) = 1$

$\xi(x, t = 0) = x$

- Solution: Link
- Mesh dependence: Link

Challenges in Higher-Dimensions

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- Velocity extrapolation needs finding the closest point to the surface.
- The advection terms $(v \cdot \nabla)(.)$ should be discretized carefully.